MATA31 Final Review

week 3 ($\lim_{x\to c} f(x) = L$

Formal definition of Limit: $\lim_{x\to c} f(x) = L$ if $\forall \varepsilon>0 \exists \delta>0 \mid x\in(c-\delta, c)\cup(c, c+\delta) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

Proof: Uniqueness of Limits.

$\boxed{\text{if } \lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} f(x) = M \text{ then } L = M}$

Proof: suppose to the contrary that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$, $L>M$, $L = M+K$

choose $\varepsilon = \frac{K}{2}$

$\lim_{x\to c} f(x) = L$ means that $\forall \varepsilon>0 \exists \delta_1>0 \mid x\in(c-\delta_1, c)\cup(c, c+\delta_1) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

$\lim_{x\to c} f(x) = M$ means that $\forall \varepsilon>0 \exists \delta_2>0 \mid x\in(c-\delta_2, c)\cup(c, c+\delta_2) \Rightarrow f(x)\in(M-\varepsilon, M+\varepsilon)$

Let δ be $\min(\delta_1, \delta_2)$, then we have

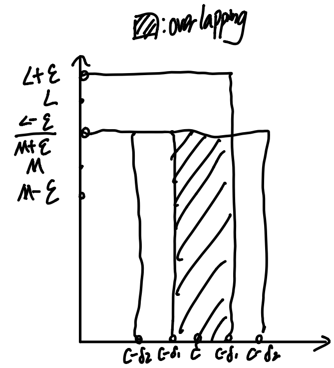
$\lim_{x\to c} f(x) = L$ means that $\forall \varepsilon>0 \exists \delta>0 \mid x\in(c-\delta, c)\cup(c, c+\delta) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

$\lim_{x\to c} f(x) = M$ means that $\forall \varepsilon>0 \exists \delta>0 \mid x\in(c-\delta, c)\cup(c, c+\delta) \Rightarrow f(x)\in(M-\varepsilon, M+\varepsilon)$

it is not possible since the intervals are not overlapping.

Therefore, the suppostion is wrong and if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$, $L = M$

QED



Left and Right limits

$\boxed{\lim_{x\to c} f(x) = L \ \boxed{\text{if and only if}} \ \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L}$

$\Longleftrightarrow$

① if $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$

② if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$ then $\lim_{x\to c} f(x) = L$

① $\lim_{x\to c} f(x) = L$ (by given) we have: $\forall \varepsilon>0 \exists \delta>0 \mid x\in(c-\delta, c)\cup(c, c+\delta) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

if $x\in(c-\delta, c) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$ then L is the left side limit.

if $x\in(c, c+\delta) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$ then L is the right side limit.

② $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$ (by given) we have: $\forall \varepsilon>0 \exists \delta_1>0 \mid x\in(c-\delta_1, c) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

$\forall \varepsilon>0 \exists \delta_2>0 \mid x\in(c, c+\delta_2) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

choose $\delta = \min(\delta_1, \delta_2)$, we have $\forall \varepsilon>0 \exists \delta>0 \mid x\in(c-\delta, c)\cup(c, c+\delta) \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

which is the definition of $\lim_{x\to c} f(x) = L$

Therefore ...

infinite limit: $\lim_{x\to c} f(x)=\infty$: $\forall M>0\ \exists\ \delta>0 \mid x\in(c-\delta,c)\cup(c,c+\delta) \Rightarrow f(x)\in(M,\infty)$

$\lim_{x\to\infty} f(x)=L$: $\forall \varepsilon>0\ \exists\ N>0 \mid x\in(N,\infty) \Rightarrow f(x)\in(L-\varepsilon,L+\varepsilon)$

$\lim_{x\to\infty} f(x)=\infty$: $\forall M>0\ \exists N>0 \mid x\in(N,\infty) \Rightarrow f(x)\in(M,\infty)$

week 4

Algebraic Definition of limit: $\lim_{x\to c} f(x)=L$ $\quad \forall\varepsilon>0\ \exists\ \delta>0 \mid 0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$

Proof ex.

① prove $\lim_{x\to 2^+} 3\sqrt{2x-4}=0$

$\forall\varepsilon>0\ \exists\ \delta>0 \mid 2<x<2+\delta \Rightarrow |3\sqrt{2x-4}|<\varepsilon$

$-\varepsilon<3\sqrt{2x-4}<\varepsilon$

$3\sqrt{2x-4}<\varepsilon \Rightarrow 3\sqrt{2}\sqrt{x-2}<\varepsilon$ $\Big\}$ $3\sqrt{2}\sqrt{\delta}=\varepsilon$

$x-2<\delta$

$\sqrt{\delta}=\frac{\varepsilon}{3\sqrt{2}}$

$\delta=\frac{\varepsilon^2}{18}$

choose $\delta=\frac{\varepsilon^2}{18}$

$3\sqrt{2x-4}=3\sqrt{2}\sqrt{x-2}<3\sqrt{2}\sqrt{\delta}=3\sqrt{2}\cdot\frac{\varepsilon}{3\sqrt{2}}$

$=\varepsilon.$

QED.

② Prove $\lim_{x\to 1}(x^2-6x+5)=0$

$\forall\varepsilon>0\ \exists\ \delta>0 \mid 0<|x-1|<\delta \Rightarrow$ $\underbrace{|x-1|}_{<\delta}\cdot\underbrace{|x-5|}_{<5}<\varepsilon \Big\}\ \delta=\frac{\varepsilon}{5}$

assume that $\delta\le 1$

$|x-1|<\delta \Rightarrow |x-1|<1$

$0<x<2$

at $x=2$ $\quad |x-5|=3$

at $x=0$ $\quad |x-5|=5$

choose δ be $\min(1,\frac{\varepsilon}{5})$

$|x^2-6x+5|=|x-1|\cdot|x-5|<5\delta=\varepsilon$

QED.

week 5

The extreme Value Theorem.

if $f(x)$ is continuous on closed interval $[a,b]$, then there exist some value M and m in $[a,b]$ such that $f(m)$ is the max value of $f(x)$ on $[a,b]$ and $f(M)$ is the min value of $f(x)$ on $[a,b]$

The Intermediate Value Theorem

if $f(x)$ is continuous on closed interval $[a,b]$, then for any K strictly between $f(a)$ and $f(b)$ there exists at least one $c\in(a,b)$ such that $f(c)=K$.

Least Upper Bound Axiom

Every nonempty set of real number that is bounded from above has a Supremum.

Greatest Lower Bound Axiom

Every nonempty set of real number that is bounded from below has a Infimum

max min: [ ]

sup Inf

sup inf: ( )

continuous: $\forall\varepsilon>0\ \exists\ \delta>0 \mid 0<|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

Proof: **The Intermediate Theorem**

Lemma: if $f(x)$ is cont on $[a,b]$, and $f(a)<0<f(b)$, then there exist such number ξ that $f(x)$ is negative on $[a,\xi)$

set $[a,\xi)$ is bounded from above by b. By Upper Bound Axiom the set has a Supremum. Let's assume that $\sup([a,\xi))=C$

$f(c)<0$ or $f(c)=0$ $\quad$ $f(c)$ cannot be greater that 0 b/c $f(x)$ is negative on $[0,\xi)$

$\downarrow$

if $f(c)<0$ then there exist number ε, $f(c\pm\varepsilon)<0$ (T.S)

if $f(c\pm\varepsilon)<0$, then $\sup([a,\xi))=c-\varepsilon$

Therefore $f(c)$ can not be negative on c and $f(c)=0$.

$$g(x)=f(x)-K$$

$g(a)=f(a)-K<0$ $\quad$ $g(b)=f(b)-K>0$, so we have conditions for the Lemma.

according to Lemma, there exist $g(c)=0$,

$$g(c)=f(c)-k=0 \Rightarrow f(c)=K$$

QED.

Constant Multiple law: $\lim\limits_{x\to c} kf(x)=k\lim\limits_{x\to c} f(x)$

Addition Law: $\lim\limits_{x\to c}[f(x)+g(x)]=L+M$

$\lim\limits_{x\to c} f(x)=L \Rightarrow \forall\varepsilon_1>0\,\exists\delta_1>0 \mid 0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon_2$

$\lim\limits_{x\to c} g(x)=M \Rightarrow \forall\varepsilon_2>0\,\exists\delta_2>0 \mid 0<|x-c|<\delta_2 \Rightarrow |g(x)-M|<\varepsilon_2$

choose $\varepsilon_1=\frac{\varepsilon}{2}$, $\varepsilon_2=\frac{\varepsilon}{2}$ and $\delta=\min(\delta_1,\delta_2)$

therefore, for $0<|x-c|<\delta$, we have $-\frac{\varepsilon}{2}<f(x)-L<\frac{\varepsilon}{2}$

$$-\frac{\varepsilon}{2}<g(x)-M<\frac{\varepsilon}{2}$$

$$-\varepsilon<f(x)+g(x)-(L+M)<\varepsilon$$

QED.

Reciprocal Law: $\lim\limits_{x\to c}\frac{1}{g(x)}=\frac{1}{M}$ $\quad M\neq 0$

Goal: $\forall\varepsilon>0\,\exists\delta>0 \mid 0<|x-c|<\delta \Rightarrow \left|\frac{1}{g(x)}-\frac{1}{M}\right|<\varepsilon$

$\left|\frac{1}{g(x)}-\frac{1}{M}\right|=\left|\frac{M-g(x)}{g(x)M}\right|$

$=\left|\frac{1}{g(x)}\right|\cdot\left|\frac{M-g(x)}{M}\right|$

Let's choose δ_1 such that $|g(x)-M|<\frac{|M|}{2} \Rightarrow$ and $|g(x)|>\frac{|M|}{2} \Rightarrow \frac{1}{|g(x)|}<\frac{2}{|M|}$

Let's choose δ_2 such that $|M-g(x)|<\frac{M^2}{2}\varepsilon \Rightarrow \left|\frac{M-g(x)}{M}\right|<\frac{M}{2}\varepsilon$

choose $\delta=\min(\delta_1,\delta_2)$

we find that if $0<|x-c|<\delta$, then $\left|\frac{1}{g(x)}-\frac{1}{M}\right|<\frac{2}{|M|}\cdot\frac{M}{2}\varepsilon=\varepsilon$

QED.

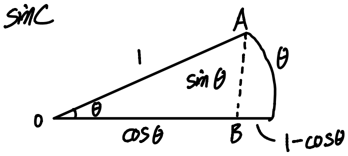
Continuity of Power Function.

$$\lim_{x\to c} x^k = c^k$$

$$\lim_{x\to c} x^k = \lim_{x\to c} x \cdot \lim_{x\to c} x \cdots \cdots$$
$$= c\cdot c\cdot c\cdot c\cdots$$
$$= c^k$$

Continuity of Trigonometric Function

We need to prove that $\lim_{x\to c} \sin x = \sin c$



① Prove $\lim_{\theta\to 0} \sin\theta = 0$

$|AB| = \sin\theta$, $0<|AB|<\theta$

$\lim_{\theta\to 0} 0 = 0$ $\quad \lim_{\theta\to 0} \theta = 0$

$0<\sin\theta<\theta$

$\lim_{\theta\to 0} \sin\theta = 0$ (By squeeze)

② prove $\lim_{\theta\to 0} \cos\theta = 1$

$0<1-\cos\theta<\theta$

$1-\theta<\cos\theta<1$

$\lim_{\theta\to 0} 1-\theta = 1$ $\quad \lim_{\theta\to 0} 1 = 1$

$\lim_{\theta\to 0} \cos\theta = 1$ (by squeeze)

Sub: let $x=c+h$

$$\lim_{x\to c} \sin x = \lim_{x\to c}(c+h) = \lim_{x\to c}(\sin c\cdot\cos h + \sin h\cdot\cos c) = \sin c\cdot 1 + 0$$
$$= \sin c$$

QED.

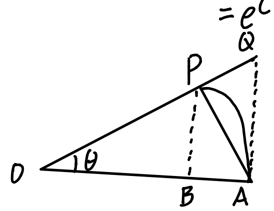
Continuity of Exponential Functions. (key: $\lim_{h\to 0} \frac{e^h-1}{h} = 1$)

$$\lim_{x\to c} e^x = e^c$$

$x = c+h$

$$\lim_{x\to c} e^x = \lim_{h\to c} e^{c+h} = \lim_{h\to 0} e^c \cdot \lim_{h\to 0} e^h = e^c \cdot \lim_{h\to 0}(e^h-1+1) = e^c\cdot \lim_{h\to 0}\left(\frac{e^h-1}{h}\cdot h+1\right)$$
$$= e^c\cdot(1) = e^c \quad \text{QED}$$

Prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$:



area of $\Delta OPA = \frac{\sin\theta}{2}$

area of $\Delta OQA = \frac{\tan\theta}{2}$

area of sector $OPA = \frac{\theta}{2}$

$$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2} \;\Big|\; \div \frac{\sin\theta}{2}$$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$$

$\lim_{\theta\to 0} 1 = 1$ $\quad \lim_{\theta\to 0} \cos\theta = 1$

$\frac{\sin\theta}{\theta} = 1$ (by squeeze).

Prove $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$$\lim_{x\to 0} \frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$
$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x} = 1\cdot\frac{0}{2} = 0$$

QED.

if $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form of $\frac{1}{0^+}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$

we need to show that $\forall M>0 \, \exists \delta>0 \mid 0<|x-c|<\delta \Rightarrow \frac{f(x)}{g(x)} > M$

we have $\lim_{x\to c} f(x) = 1$: $\forall \varepsilon_1 > 0 \, \exists \delta_1 \mid 0<|x-c|<\delta_1 \Rightarrow |f(x)-1|<\varepsilon_1$ $\quad 1-\varepsilon_1 < f(x) < \varepsilon_1+1$
(no > 0)

$\lim_{x\to c} g(x) = 0^+$: $\forall \varepsilon_2 > 0 \, \exists \delta_2 \mid 0<|x-c|<\delta_2 \Rightarrow |g(x)|<\varepsilon_2$ $\quad -\varepsilon_2 < g(x) < \varepsilon_2$
(no > 0)

$\frac{f(x)}{g(x)} > \frac{1-\varepsilon_1}{\varepsilon_2}$ $\quad$ let $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \frac{1}{2M}$

$\frac{f(x)}{g(x)} > \frac{1-\frac{1}{2}}{\frac{1}{2M}} = M$ QED.

if $\lim_{x\to\infty}$ is of the form $\frac{1}{\infty}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

we need to show that $\forall \varepsilon > 0 \, \exists N>0 \mid x>N \Rightarrow \left|\frac{f(x)}{g(x)}\right| < \varepsilon$

$\lim_{x\to\infty} f(x) = 1$: $\forall \varepsilon_1 > 0 \, \exists N_1 > 0 \mid x > N_1 \Rightarrow |f(x)-1| < \varepsilon_1$
$1-\varepsilon_1 < f(x) < \varepsilon_1 + 1$

$\lim_{x\to\infty} g(x) = \infty$: $\forall M_1 > 0 \, \exists N_2 > 0 \mid x > N_2 \Rightarrow g(x) > M$

$\frac{f(x)}{g(x)} < \frac{\varepsilon_1+1}{M}$, Let $\varepsilon_1 = 1$ $\quad M = \frac{2}{\varepsilon}$

$\therefore \frac{f(x)}{g(x)} < \frac{1+\varepsilon_1}{M} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon$ QED.

week 8 (Derivative)

Constant function: $\frac{d}{dx}(c)=0$

$(c)'=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{c-c}{h}=0$ QED

The power Rule: $f(x)=x^n,\ f'(x)=nx^{n-1}$

$(x^n)'=\lim_{h\to0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to0}\frac{x^n+nx^{n-1}\cdot h+n\frac{n-1}{2}x^{n-2}h^2+\dots h^n-x^n}{h}$

$=\lim_{h\to0}\frac{nx^{n-1}\cdot h+n\frac{n-1}{2}x^{n-2}h^{2}+\dots+h^n}{h}$

$=nx^{n-1}$ QED

The constant Multiple Rule: $(rf')(x)=rf'(x)$

$(rf)'=\lim_{h\to0}\frac{rf(x+h)-rf(x)}{h}=r\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=rf(x)$

QED.

Sum Rule: $(f+g)'(x)=f'(x)+g'(x)$

$(f+g)'(x)=\lim_{h\to0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to0}\frac{g(x+h)-g(x)}{h}$

$=f'(x)+g(x)$

QED.

Produce Rule: $(f\cdot g)'(x)=f'(x)g(x)+f(x)g'(x)$

$(f\cdot g)'(x)=\lim_{h\to0}\frac{f(x+h)\cdot g(x+h)-f(x)g(x)-f(x)g(x+h)+f(x)g(x+h)}{h}$

$=\lim_{h\to0}\frac{g(x+h)[f(x+h)-f(x)]+f(x)[g(x+h)-g(x)]}{h}$

$=f'(x)g(x)+g'(x)f(x)$ QED

Quotient Rule: $\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

$\left(\frac{f(x)}{g(x)}\right)'=\left(f(x)\cdot g^{-1}(x)\right)'=f'(x)\cdot g^{-1}(x)+f(x)[g^{-1}(x)]'$

$=\frac{f'(x)}{g(x)}+f(x)[-g^{-2}(x)\cdot g'(x)]$

$=\frac{f'(x)}{g(x)}-\frac{f(x)\cdot g'(x)}{g^2(x)}$

$=\frac{f'(x)g(x)}{g^2(x)}-\frac{f(x)\cdot g'(x)}{g^2(x)}$

$=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$ QED.

Chain Rule: $(f\circ g)'(x)=f'(g(x))\cdot g'(x)$

$(f\circ g)'(x)=f'(g(x))$

$=\lim_{h\to0}\frac{f(g(x+h))-f(g(x))}{h}\cdot\frac{g(x+h)-g(x)}{g(x+h)-g(x)}$

$=\lim_{h\to0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}$

$=\lim_{k\to0}\frac{f(g(x)+k)-f(g(x))}{k}\cdot g'(x)$

$=f'(g(x))\cdot g'(x)$

QED.

Exponential Function: $f(x)=a^x,\ f'(x)=a^x\ln a.$

$(a^x)'=(e^{x\ln a})'$

$=e^{x\ln a}\cdot(x\ln a)'=e^{x\ln a}\cdot\ln a=a^x\cdot\ln a$

QED

Logarithmic Functi

a) $f(x)=\log_b x$, $f'(x)=\frac{1}{x\ln b}$

$x=b^{\log_b x}$

$(x)'=(b^{\log_b x})'$

$1=b^{\log_b x}\cdot\ln b\cdot(\log_b x)'$

$(\log_b x)'=\frac{1}{x\ln b}$

b) $f(x)=\ln x$

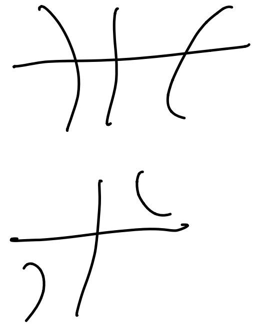
$(e^{f(x)})'=(e^{\ln x})'$

$e^{f(x)}\cdot f'(x)=x'$

$x\cdot f'(x)=1$

$f'(x)=\frac{1}{x}$ QED.

c)



$[\ln(x)]'=\frac{1}{x}$

Trigonometric Functin:

$(\cos x)'=-\sin x$

$$(\cos x)'=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{\cos x\cosh-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{\cos x(\cos h-1)}{h}-\lim_{h\to 0}\sin x\frac{\sin h}{h}$$

$=0-\sin x$

$=-\sin x$

QED.

Hyperbolic Functin:

$\cosh x=\frac{e^x+e^{-x}}{2}$ $\sinh x=\frac{e^x-e^{-x}}{2}$

$$(\cosh x)'=\left(\frac{e^x+e^{-x}}{2}\right)'=\frac{1}{2}(e^x-e^{-x})=\frac{e^x-e^{-x}}{2}=\sinh x$$

$\sin^2 x + \cos^2 x = 1$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \pm \sin x \sin y$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Trigonometric function derivative

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\csc x)' = -\csc x \cot x$

$(\sec x)' = \sec x \tan x$

$(\cot x)' = -\csc^2 x$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $(\arctan x)' = \frac{1}{1+x^2}$ $(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$ $(\operatorname{arccsc} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$

$(\sinh x)' = \cosh x$ $(\cosh x)' = \sinh x$ $(\tanh x)' = \operatorname{sech}^2 x$.

$(\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$ $(\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$

$(\coth x)' = -\operatorname{csch}^2 x$

$(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$ $(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}$ $(\tanh^{-1} x)' = \frac{1}{1-x^2}$

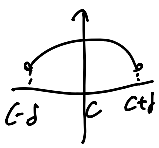
week 10

Proof — **Fermat's Theorem**: if $f(x)$ has a local extremum at an interior point c and $f'(x)$ exist, then $f'(c)=0$

① Given local max of $f(x)$ at $x=c$

$f(c) \geq f(x)$

$x \in (c-\delta, c+\delta)$



$$\left.\begin{array}{l} x \in (c, c+\delta) \\ f(x)-f(c) \leq 0 \\ x-c \geq 0 \end{array}\right\} f'(c^+) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$$

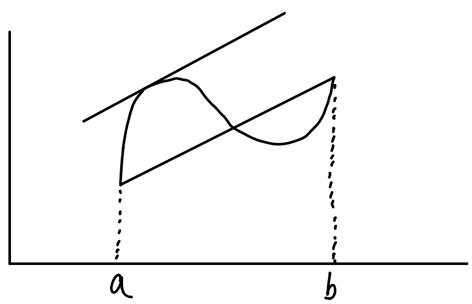
$$\left.\begin{array}{l} x \in (c-\delta, c) \\ f(x)-f(c) \leq 0 \\ x-c \leq 0 \end{array}\right\} f'(c^-) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$$

② "$f'(c)$ does exist" means that $f'(c^-)=f'(c^+)$

But $f'(c^-)=f'(c^+)$ iff $f'(c^-)=0$ and $f'(c^+)=0$

$\therefore f'(c)=0$ QED.

Mean value theorem: if $f(x)$ is cont on $[a,b]$ and diff (a,b) then $\exists$ at least one number $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$



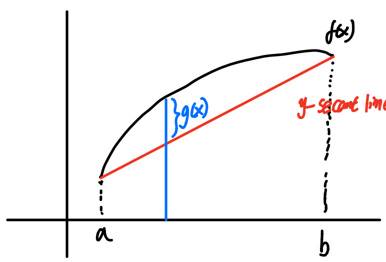
$\frac{f(b)-f(a)}{b-a}$ - average rate of change $[a,b]$

$$(\log_b x)' = \frac{1}{x \ln b}$$

$$x = b^{\log_b x}$$

$$1 = b^{\log_b x}$$

Proof:



Let $g(x)=f(x)-y(x)$

$$\left.\begin{array}{l} g(x) - \text{diff on } (a,b) \\ g(x) - \text{con on } [a,b] \\ g(a)=f(a)-y(a)=0 \\ g(b)=f(b)-y(b)=0 \end{array}\right\} \begin{array}{l} \text{By Rolle's Theorem} \\ \exists\, c \in (a,b) : g'(c)=0 \end{array}$$

Point-Point Eq of Secant line

$$\frac{x-a}{b-a} = \frac{y-y(a)}{y(b)-y(a)} \Rightarrow y-y(a) = \frac{y(b)-y(a)}{b-a}(x-a)$$

$$y = y(a) + \underbrace{\frac{y(b)-y(a)}{b-a}}_{\text{slope}}(x-a)$$

from $g(x)=f(x)-y(x)$ we have $f'(x)=g'(x)+y'(x) \to$ slope

$$f'(c) = g'(x) + \frac{f(b)-f(a)}{b-a} = 0 + \frac{f(b)-f(a)}{b-a}$$

QED

Squeeze Theorem:

if $f(x) \le g(x) \le h(x)$, for all x and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$$

then

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

slant asyptote

i) $\dfrac{Q_n(x)}{P_m(x)}$ $n > m$

HA $n = m$

only VA $m > n$

ii) log or $\sqrt{2}$ or hyperbolic

RSL: $y = k_1x + b_1$

$$k_1 = \lim_{x\to\infty} \frac{f(x)}{x} \quad b_1 = \lim_{x\to\infty} [f(x) - k_1x]$$

LSL: $y = k_2x + b_2$

$$k_2 = \lim_{x\to-\infty} \frac{f(x)}{x}, \quad b_2 = \lim_{x\to-\infty} [f(x) - k_2x]$$

Newton's Method

$$\lim_{x\to0} (1+x)^{\frac{1}{x}} = e$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\lim_{h\to0} \frac{e^h - 1}{h} = 1$$

Bonus:

① Prove $(\arctan x)' = \dfrac{1}{1+x^2}$

$y = \arctan x$

$x = \tan y$

$(x)' = (\tan y)'$

$1 = \sec^2 y \cdot y'(x)$

$y'(x) = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$

since $\tan y = x$
$\tan^2 y = x^2$

Prove: $(\text{arcsec}\, x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$

$y = \text{arcsec}\, x \Leftrightarrow \text{acrsec}\, y = x$

① $y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$(x)' = \sec y \cdot \tan y \cdot y'(x)$

$y'(x) = \dfrac{1}{\sec y \cdot \tan y}$

on $[0, \frac{\pi}{2})$ $\tan y > 0$ $\tan y = \sqrt{\sec^2 - 1}$

on $[\pi, \frac{3\pi}{2})$ $\tan y > 0$ $\tan y = \sqrt{\sec^2 - 1}$

$y' = \dfrac{1}{\sec y \cdot \sqrt{\sec^2 - 1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$

② $y \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

$(x)' = \sec y \cdot \tan y \cdot y'(x)$

$y'(x) = \dfrac{1}{\sec y \cdot \tan y}$

on $[0, \frac{\pi}{2})$ $\tan y > 0$ $\tan y = \sqrt{\sec^2 - 1}$, $y' = \dfrac{1}{x\sqrt{x^2-1}}$

on $(\frac{\pi}{2}, \pi]$ $\tan y > 0$ $\tan y = -\sqrt{\sec^2 - 1}$, $y = \dfrac{1}{-x\sqrt{x^2-1}}$

$\therefore$ $y' = \dfrac{1}{|x|\sqrt{x^2-1}}$